RADIATIVE EVOLUTION OF ORBITS AROUND

A KERR BLACK HOLE

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Abstract

We propose a simple approach for the radiative evolution of generic orbits around a Kerr black hole. For a scalar-field, we recover the standard results for the evolution of the energy E and the azimuthal angular momentum L_z . In addition, our method provides a closed expression for the evolution of the Carter constant Q.

Recently, the prospects of a direct observation of gravitational radiation led to a new interest in the old-standing problem of radiation back-reaction. The most promising radiation sources for the proposed gravitational-waves detector LIGO (as well as LISA [1]) are coalescing binary systems. [2] In such a binary system, energy and angular momentum are carried away by the emitted gravitational radiation. As a consequence, the system becomes more and more bound, resulting in a steady grow in the orbital frequency and energy-loss rate. Eventually, the binaries will coalless, releasing a strong burst of gravitational radiation. [2] The prospects of observing such bursts raises the need for a theoretical prediction of the evolution of such binary systems. If one of the binary objects is a black hole, then curvature effects could be important. This motivates one to understand the radiation-reaction in curved spacetime.

Consider the situation of a compact object with mass μ orbiting a super-massive Kerr black hole with mass $M >> \mu$ and angular momentum aM (we are using general-relativistic units C=G=1). This situation is particularly important for the proposed space-based gravitational-waves detector LISA, [1,3] but is also interesting in its own right. The problem of radiative evolution of orbits around a Kerr black hole was treated by several authors [4, 5], but so far with partial success only. The orbits in Kerr geometry are characterized by three constants of the motion: the energy E , the azimuthal angular momentum L_z , and the Carter constant Q . [6] Within the adiabatic approximation [3] that we use here, the problem of radiative evolution is reduced to that of calculating $\dot{E}$, $\dot{L}_z$ and $\dot{Q}$, where the overdot denotes the evolution rate (in terms of external time t). So far, expressions have been given for $\dot{E}$ and $\dot{L}_z$, based on the rate that energy and angular-momentum are carried away to infinity and into the black hole [4,5,7]. However, $\dot{Q}$ is still unknown. [8]

Although we are primarily interested in the backreaction of gravitational radiation, it would be convenient to generalize the discussion and to consider electromagnetic and scalar-field radiation as well. We shall thus assume that the particle carries a "generalized charge" q , which may be either a scalar, electromagnetic, or gravitational charge (we demand $q << M$; in the gravitational case, $q \equiv \mu$). In the discussion below we shall mainly refer to scalar and electromagnetic radiation (the explicit calculations in this Letter will be restricted to the scalar-field case) ; However, our approach can be extended to gravitational radiation as well [9].

Our main goal here is to propose a scheme for the calculation of $\dot{Q}$ (as well as $\dot{E}$ and $\dot{L}_z$) for generic orbits in Kerr spacetime, by a direct calculation of the radiation-backreaction force - based on the pure retarded potential. An analogous direct scheme for radiative evolution of E and L_z has already been proposed by Gal'tsov [4] , but that scheme is based on the so-called "radiative potential" (the "half- retarded-minus-half-advanced" potential). This effective potential was devised by Dirac [10] in order to handle the divergencies encountered in calculations of the radiation-reaction force in flat spacetime. We point out, however, that in general the radiative potential will not be valid in curved spacetime. Evidently, this potential yields non-causal results: Due to the scattering off the curvature, the Green's functions of the massless fields will have a support not only on the light-cone itself, but also in its interior. Thus, with the advanced potential, the backreaction force acting on the particle at a given moment will in general depend on the particle's future history. Clearly, the radiative potential will inherit this non- causal behavior, indicating that in general this potential is not valid in curved spacetime. Although the radiative potential yields the correct results for $\dot{E}$ and $\dot{L}_z$ in Kerr spacetime [4] , it is not clear whether the corresponding results for $\dot{Q}$ will be correct too.

The inapplicability of the retarded potential in curved spacetime motivates us to base our approach directly on the retarded potential. For concreteness and for conceptual clarity, let us refer at this stage of the discussion to the electromagnetic radiation reaction (later we shall generalize this discussion to a scalar field and implement it for explicit calculations). Consider an electrically-charged particle that moves in curved spacetime. In principle, the backreaction force has a simple origin: It is just the Lorentz force, experienced by the particle due to its interaction with its own electromagnetic field. However, the field of a point-like particle is divergent at the particle's location, and any attempt to calculate the backreaction force must first address this conceptual difficulty. In flat space, the radiative potential is often used in order to overcome this difficulty. Unfortunately, this method is not valid in curved spacetime. A fully-relativistic analysis of electromagnetic radiation-reaction in curved spacetime was carried out by DeWitt and Brehme [11]. This analysis is based on the concept of a world-tube (with finite radius ε) surrounding the charged particle. By taking the limit $\varepsilon \to 0$, DeWitt and Brehme obtained a closed expression for the electromagnetic backreaction force. However, this expression is not so useful for explicit calculations: It includes a "tail" term, given by an integral over proper time along the particle's worldline. It is hard to calculate this tail-integral explicitly - especially because one does not know the explicit form of the Green's function that appears in the integrand.

Our approach to the problem is based on the following observation: Although the particle's self field is divergent, the modes of the field (e.g. the spin-weighted spherical harmonics for Schwarzschild gemetry, and the spin-weighted spheroidal harmonics for Kerr gemetry) are everywhere finite and well-defined - even at the limit of a point-like source. Thus, one may simply calculate the contribution of each mode to the radiative evolution (by a direct application of the Lorentz-force formula to the interaction of the particle with the mode in question), and then sum over the modes. By this decomposition into modes we achieve two goals: First, we remove the singularity of the self-field. [1] Second, this is the only practical way to solve the field equations for a Kerr background.

[1] The contribution of each mode to the radiation-reaction is guaranteed to be finite. One still may be concerned about the convergence of the sum over the modes. At least for $\dot{E}$ and $\dot{L}_z$, this sum is finite. See the discussion at the end of the paper.

We still have to address the issue of renormalization. It is well known, already from flat space, that in general the retarded backreaction force must be renormalized before it can be used for explicit radiation-reaction calculations. The same phenomenon occurs for charged particles in curved spacetime. [11] (Indeed, it was this complication which motivated Gal'tsov [4] to use the radiative potential). Fortunately, it turns out that in the problem considered here no such force-renormalization is needed: The only need (and justification) for the force-renormalization originates from the renormalization of the mass-parameter. It therefore follows that, from its very nature and definition, the force-renormalization must take the form

$$F^{\alpha}_{eff} = F^{\alpha}_{ret} + \mu_{se} a^{\alpha} \; , \tag{1}$$

where F^{α}_{ret} is the retarded backreaction force, F^{α}_{eff} is the effective (i.e. renormalized) force, a^{α} is the covariant four-acceleration, and μ_{se} is the parameter that renormalizes the particle's mass (the particle's "self-energy"). This argument [and Eq. (1)] holds in both flat and curved spacetimes. [2] In the problem discussed here, the radiative evolution of orbits around a Kerr black hole, no external force is considered, so (within the framework of the adiabatic approximation) $a^{\alpha} = 0$ and therefore $F^{\alpha}_{eff} = F^{\alpha}_{ret}$. [3]

We turn now to the concrete calculations of the radiative evolution of generic orbits in Kerr spacetime. For simplicity, we shall focus here on (massless, minimally-coupled) scalar-field radiation reaction. The generalization of this calculation scheme to electromagnetic and gravitational radiation is outlined in Ref. [9]. Let C denote the constant in question - E , L_z or Q . All these constants may be expressed explicitly as functions of coordinates and (covariant) four-velocity, i.e. $C = C(x^{\beta}, u_{\alpha})$. Thus, we may write [6]

$$E = -u_t \quad , \quad L_z = u_{\varphi} \quad , \quad Q = u_{\theta}^2 + \cos^2\theta \, [a^2(1-u_t{}^2) + \sin^{-2}\theta \, u_{\varphi}^2] \; .$$

(We use Boyer-Lindquist coordinates.) The force induced by a scalar field $\psi(x^{\alpha})$ on a particle with a scalar charge q is given by

$$F_{\alpha} = q(\psi_{,\alpha} + u_{\alpha} u^{\beta} \psi_{,\beta}) \equiv q \psi_{:\alpha} \; ; \tag{2}$$

this is the scalar-field analog of the electromagnetic Lorentz force [4] . [At this stage, $\psi(x^{\alpha})$ may be thought of as any prescribed scalar field; We shall soon apply this formalism to the backreaction problem.] It is not difficult to show [9] that the parameters C evolve according to

$$\frac{dC}{d\tau} = C^{\alpha} F_{\alpha} = q C^{\alpha} \psi_{:\alpha} \; , \tag{3}$$

where τ is the particle's proper time and $C^{\alpha} \equiv \mu^{-1} \, \partial C / \partial u_{\alpha}$.

Next, we decompose ψ into its Teukolsky modes [12] :

$$\psi = \sum_{lm\omega} \psi^{lm\omega} \quad , \quad \psi^{lm\omega} = R_{lm\omega}(r) S_{lm\omega}(\theta) e^{i(m\varphi - \omega t)} \; . \tag{4}$$

From Eqs. (2,3), the contribution of each mode to the backreaction force and to $dC/d\tau$ is

$$F^{lm\omega}_{\alpha} = q \psi^{lm\omega}_{:\alpha} \quad , \quad \left(\frac{dC}{d\tau}\right)_{lm\omega} = q C^{\alpha} \psi^{lm\omega}_{:\alpha} \; , \tag{5}$$

where

$$\psi^{lm\omega}_{:\alpha} \equiv \psi^{lm\omega}_{,\alpha} + u_{\alpha} u^{\beta} \psi^{lm\omega}_{,\beta} \; .$$

The contribution of each mode l, m, ω to the change in C is thus

$$\Delta C_{lm\omega} = q \int C^{\alpha} \psi^{lm\omega}_{:\alpha} \, d\tau \; . \tag{6}$$

[2] Note the consistency of Eq. (1) with the results obtained in Ref. [11] for electromagnetic radiation reaction in curved spacetime; See in particular Eq. (5.23) there.

[3] This statement can be made more precise in terms of the world- tube approach of Ref. [11] . For any finite radius ε , the self-energy term in the backreaction force is well-defined [cf. Eq. (5.23) there] - and vanishes for $a^{\alpha} = 0$. One can then safely take the limit $\varepsilon \to 0$.

In this expression, the integrand is to be evaluated along the particle's worldline $x^\alpha(\tau)$.

In order to apply Eq. (6) to the backreaction problem, we take $\psi^{lm\omega}$ to be the retarded-field modes associated with the charged particle itself. To determine these modes, one decomposes the particle's charge-density function $\rho(x^\alpha)$ into its modes according to [12]

$$4\pi\Sigma\rho = \sum_{lm\omega} T_{lm\omega}(r) S_{lm\omega}(\theta) e^{i(m\varphi-\omega t)} \quad , \tag{7}$$

where $\Sigma \equiv r^2 + a^2\cos^2\theta$. The radial functions $R_{lm\omega}(r)$ are then determined by the Teukolsky equation [12]

$$(\Delta\, R_{lm\omega}) \;+ V_{lm\omega}(r) R_{lm\omega} = -T_{lm\omega} \tag{8}$$

(with retarded boundary conditions), where $\Delta \equiv r^2 - 2Mr + a^2$, a prime denotes d/dr, and $V_{lm\omega}(r)$ is some real potential. We take the charge-density function $\rho(x^\alpha)$ to be that of a point-like particle:

$$\rho(x^\alpha) = q\sqrt{-g}^{\,-1} \int \delta\left(x^\alpha - x^\alpha(\tau)\right) d\tau \quad , \tag{9}$$

where $x^\alpha(\tau)$ denotes the particle's worldline. (This approximation demands that the particle's radius d will satisfy $d << M$, which we shall assume.) Note that the thus-obtained radial functions $R_{lm\omega}(r)$ are everywhere smooth (C^1). [4]

Finally, the long-term evolution rate (in terms of asymptotic time t) is given by

$$\dot{C} = \sum_{lm\omega} \dot{C}_{lm\omega} \quad , \quad \dot{C}_{lm\omega} = \Delta t^{-1} Real\left(\Delta C_{lm\omega}\right) \quad . \tag{10}$$

(The imaginary part of $\Delta C_{lm\omega}$ will cancel out upon summation, because $\Delta C_{l,-m,-\omega} = \Delta C^*_{lm\omega}$ [9].) Here, $\Delta t \equiv t(\tau_2) - t(\tau_1)$, where τ_1 and τ_2 are the lower and upper integration limits in Eq. (6). For strictly periodic orbits, we choose τ_1 and τ_2 such that $\Delta\tau \equiv \tau_2 - \tau_1$ is the orbital period; For the generic orbits in Kerr spacetime, which are not strictly periodic but rather quasi- periodic, we take $\Delta\tau$ to be sufficiently large, so that the quasi-periodicity will yield a well-defined limit in Eq. (10) (formally speaking, for quasi-periodic orbits we shall be interested in the limit $\Delta\tau \to \infty$).

Equations (6-10) constitute a closed scheme for the computation of the (scalar-field) radiative evolution of all constants of the motion in Kerr spacetime. The calculation may be further simplified, however, by using Eq. (9) to rewrite Eq. (6) as

$$\Delta C_{lm\omega} = \int C^\alpha \psi^{lm\omega}_{:\alpha} \;\Sigma\rho\, \sin\theta\, d^4x^\alpha \tag{11}$$

(recall $\sqrt{-g} = \Sigma\sin\theta$). [5] We shall now treat the cases $C = E$ and $C = L_z$, which are especially simple. Let P denote either E or L_z , and let p denote ω or m , correspondingly. From Eq. (4) and the definitions of C^α and C one then finds (ignoring terms which just oscillate in time)

$$\Delta P_{lm\omega} = \frac{ip}{\mu} \int \psi^{lm\omega} \;\Sigma\rho\, \sin\theta\, d^4x^\alpha \quad . \tag{12}$$

Finally, using the decompositions (4,7), and the orthogonality of the modes, one obtains

$$\dot{P}_{lm\omega} = \frac{ip}{4\mu} \int R^{lm\omega}\,(r) T^*_{lm\omega}\,(r) dr + C.C. \quad . \tag{13}$$

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[4] The special case of circular orbits (r=const) is exceptional, as the radial functions are not C^1 at the particle's location. This simpler case has to be treated separately.

[5] C^α is only defined along the particle's worldline; However, the integrand in Eq. (11) vanishes elsewhere anyway.

It would be interesting to compare Eq. (13) to the standard expressions for energy and angular-momentum radiative losses. By virtue of Eq. (8), it is possible to integrate Eq. (13) explicitly:

$$\dot{P}_{lm\omega} = \frac{ip}{4\mu}\left[\Delta\, R^{*lm\omega}\, R^{lm\omega}_{,r}\right]_{r_+}^{\infty} + C.C. \ , \tag{14}$$

where r_+ denotes the horizon's radius (this equation may easily be checked by a direct differentiation). Taking into account the asymptotic behavior of the radial functions in both asymptotic limits [7] , it is now straightforward to show that Eq. (14) is exactly of the desired form; Namely, $\dot{P}_{lm\omega}$ is the sum of two terms - the fluxes (per unit mass μ) of energy or azimuthal angular momentum that flow (i) to infinity and (ii) into the horizon - exactly as calculated by the standard method [7] .

The more interesting application of our scheme would be the calculation of $\dot{Q}$ (here there are no previous results to compare with). Equations (10-11) provide a closed scheme for the calculation of $\dot{Q}$. In Ref. [9], however, we derive from Eqs. (10-11) a more explicit expression for $\dot{Q}_{lm\omega}$:

$$\dot{Q}_{lm\omega} = \mu^{-1}\int\left(HR^{lm\omega}\,T^{*}_{lm\omega} - R^{lm\omega}\,\tilde{T}^{*}_{lm\omega} - R^{lm\omega}_{,r}\hat{T}^{*}_{lm\omega}\right)dr + C.C. \ , \tag{15}$$

where

$$H \equiv ir\Delta^{-1}\left[\omega E(r^3+ra^2+2Ma^2) - 2(\omega L_z + mE)Ma + mL_z(2M-r)\right]$$

and $\tilde{T}_{lm\omega}(r)$ and $\hat{T}_{lm\omega}\,(r)$ are functions analogous to $T_{lm\omega}$, obtained by replacing the factor ρ at the left-hand side of Eq. (7) with $\tilde{\rho} \equiv 2ru^r\rho$ or $\hat{\rho} \equiv \Delta u_r\,\rho$, correspondingly. Equation (15) thus allows a straightforward numerical calculation of $\dot{Q}_{lm\omega}$ (and $\dot{Q}$).

We wish to emphasize that no attempt was made here to provide a complete and fully-rigorous treatment of all the subtle issues involved in the radiation-reaction problem. Rather, the goal here was to propose a simple and straightforward approach to the problem, which appears to yield promising results: It recovers the standard results for $\dot{E}$ and $\dot{L}_z$ in Kerr spacetime (at least for a scalar field) - and also yields an explicit expression for $\dot{Q}$. Our approach is based on the delta- function approximation, Eq. (9). This approximation has extensively been used over the years for radiation-reaction calculations (see for example Refs.[3-5] and references therein). We believe that this approximation is justified in our case. We hope that in the future it will be possible to apply more rigorous methods for the calculation of the radiative evolution of orbits in Kerr spacetime.

One may be concerned about the convergence of the sum over the modes in Eq. [10] . This series is known to converge for $C = E$ and $C = L_z$, and this makes us optimistic about its convergence for $C = Q$ as well. The explicit calculation scheme presented here allows one to verify this convergence (for $Q = C$) numerically - and perhaps even analytically.

Finally, we mention several tests which may be applied to our result for $\dot{Q}$, Eq. (15). The first such test would be, of course, to verify converges of the sum over the modes (see above). Another simple test can be applied to the special case a=0 (the Schwarzschild case): Due to the spherical symmetry of the latter, the orbital plane must be preserved. A simple calculation then yields $\dot{Q} = 2(Q/L_z)\,\dot{L}_z$. Although this test is limited to the case a=0 , it is conceivable that if our scheme is faulty, it will yield wrong results in that case too. Finally, for equatorial orbits ($Q = 0$) in Kerr spacetime, $\dot{Q}$ must vanish identically.

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